The Topology of the Regularized Integral Surfaces of the 3-Body Problem

by

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1. Introduction

It is known from the work of Levi-Civita and Sundman that singularities due to binary collisions in the 3-body problem can be regularized. That is given a solution γ which "ends" in a binary collision at time t_0 one can make a change of phase space variables and time scale such that the new variables can be continued as convergent power series in the new time for times beyond the collision date. This result is satisfactory from the analytical point of view, but from the qualitative point of view one would like to know the phase portrait of solutions near collisions.

A first step in the qualitative study of the 3-body problem might be to study the topology of the integral surfaces; that is, the momentum, angular momentum and energy. This has been done by Smale [10] and Easton [7]. These surfaces are not compact and some solutions may "run off" these surfaces in finite time and hence Newton's equations of motion do not give rise to flows on these integral surfaces One might guess that the bad behavior of solutions is due to collisions of the bodies. It is well known that if the total angular momentum of the 3-bodies is different from zero then a simotaneous collision of all three bodies is impossible. Thus on integral surfaces with non-zero angular momentum one expects the only bad behavior of solutions will be due to binary collisions. In view of the result that solutions can be continued through binary collisions one might hope to modify the integral surface in some way so that Newton's equations of motion actually give a flow. The purpose of the present paper is to describe how this can be done.

One might ask why one should study collisions in the first place. The set of solutions which end in collisions is known to have measure zero and one might argue that hence it can be neglected. My answer is that the set of solutions which pass close to collisions does not have measure zero and that these solutions can be conveniently studied by focusing attention on those solutions which do end in collision. Furthermore one can not decide a priori whether or not a given initial condition will or will not eventually lead either to collision or to a very close approach to collision. Hence the full phase portrait of the solutions can not be understood without knowing what happens near collision.

The techniques used in the present paper were introduced in [6]. We use surgery to exercise a neighborhood of the "binary collision set" (see definition 3.1). The neighborhood is in the form of an "isolating block" (see definition 2.3). We identify the end points of orbits which cross the block and we show that this identification has a unique extension to an identification which pairs the end points of orbits entering the block which end in a binary collision with the end points of orbits leaving the block which come from a binary collision. The problem of regularization is the problem of showing that the identification of the end points of crossing orbits has a continuous, unique extension. We use this identification to close the gap left by the surgery thus obtaining the "regularized" phase space for the 3-body problem. We obtain regularized integral surfaces for the problem on which the 3-body equations of motion induce flows. Finally we describe the topology of these surfaces thus answering a question for the planar 3-body problem raised by Birkhoff [1, p. 288] and again by Wintner [11, Section 438].

Throughout the paper we restrict our attention to the planar 3-body problem. The extension of some of our results to the non-planar problem may not be trivial.

C. Conley has shown that tripple collisions can not be regularized by surgery. A partial discussion of this result is given in section 6.

2. Regularization by Surgery

Regularization of vector fields by surgery is discussed in [6]. We will give in this section a brief description of this process. Let M be a C^∞ manifold and let C be a closed subset of M.

We assume throughout this section that X is a C^∞ vector field defined on $M-C$. We call C the "singularity" of the vector field.

Notation 2.1: If $a \in M-C$ and $\gamma(t)$ is an integral curve of X satisfying $\gamma(0) = a$ we denote $\gamma(t)$ by the notation $\gamma(t) = a\cdot t$. More generally if $A \subset M-C$ and $T \subset R^1$ and if $a\cdot t$ is defined for each pair $(a,t) \in A \times T$ we let

$$A\cdot T = \{a\cdot t\colon a \in A \text{ and } t \in T\}.$$

Definition 2.2: Suppose that $F\colon M-C \to R^1$ is a smooth function and define $\dot{F}\colon M-C \to R^1$ by $\dot{F}(a) = \frac{d}{dt} F(a\cdot t)\big|_{t=0}$. Also define $\ddot{F} = \dot{G}$

where $G = \dot{F}$.

Definition 2.3: Let $F_j: M-C \to R^1$ be a smooth function for $j = 1,\ldots,k$ and let

$$B = \{x \in M-C: F_j(x) \leq 0 \text{ for } j = 1,\ldots,k\}.$$

B is an isolating block for X if for each point $x \in B$, whenever $F_j(x) = 0$, and $\dot{F}_j(x) = 0$, it is the case that $\ddot{F}_j(x) > 0$.

Isolating blocks have been studied in [2], [3], [4], [5] and the above definition is not the most general that could be given. Many examples of isolating blocks and the uses of isolating blocks are given in the papers cited above. The following two examples illustrate how they may naturally occur.

Let N be a compact Riemanian manifold and let G be a Morse function $G: N \to R^1$. Let $Y = \text{grad } G$. Then if c_1 and c_2 are two non-critical values of G the set $G^{-1}[c_1,c_2]$ is an isolating block for the vector field Y on N. If N is the torus and G is the standard height function on the torus then the shaded region shown in figure 1 is an isolating block for the gradient flow.

Consider the vector field Z on R^2 given by $\dot{x} = x$, $\dot{y} = -y$ and let $F_1(x,y) = 1-x^2$ and $F_2(x,y) = 1-y^2$. Then $B = \{(x,y): F_1(x,y) \leq 0 \text{ and } F_2(x,y) \leq 0\}$ is an isolating block for Z as shown in figure 1.

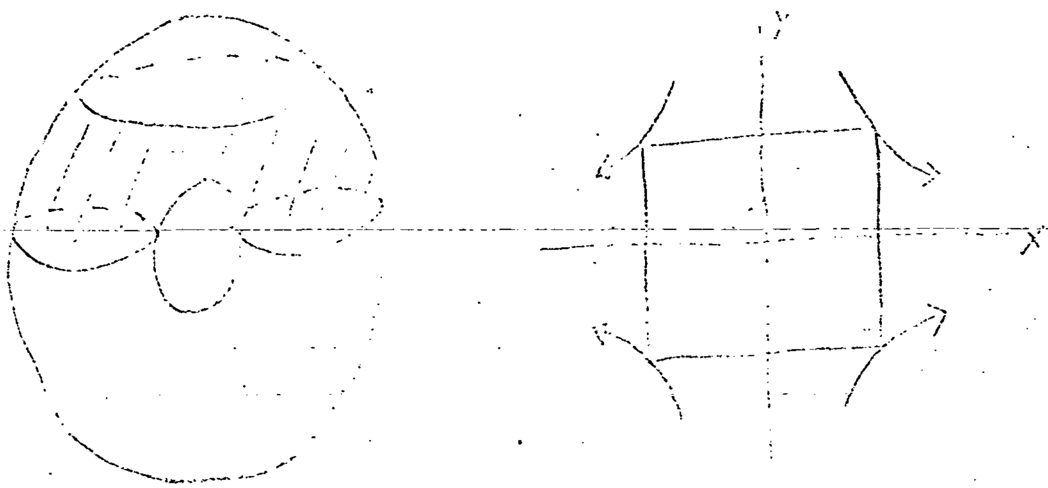

Figure 1

<u>Definition 2.4</u>: Let B be as in 2.3 and define $b = \partial B$

$$b^+ = \{x \in B: x\cdot(-\epsilon,0) \cap B = \phi \text{ for some } \epsilon > 0\}$$

$$b^- = \{x \in B: x\cdot(0,\epsilon) \cap B = \phi \text{ for some } \epsilon > 0\}$$

$$A^+ = \{x \in B: x\cdot t \in B \text{ for all } t \geq 0 \text{ for which } x\cdot t \text{ is defined}\}$$

$$A^- = \{x \in B: x\cdot t \in B \text{ for all } t \leq 0 \text{ for which } x\cdot t \text{ is defined}\}$$

$$a^+ = A^+ \cap b \text{ and } a^- = A^- \cap b$$

Define π: $b^+ - a^+ \to b^- - a^-$ by setting

$$\pi(x) = x\cdot\sigma \text{ where } \sigma = \sup\{t \geq 0: x\cdot t \in B\}.$$

<u>Theorem 2.5</u>: π is a homeomorphism.

For a proof see [4] or [5]. This theorem is what makes isolating blocks useful. An isolating block must certainly isolate something and the following discussion says what that is.

Definition 2.6: A closed set $I \subset M\text{-}C$ is called an invariant set of X if $I \cdot R^1$ is defined and if $I \cdot R^1 = I$. An invariant set I of X is said to be isolated if there exists an open set U containing I such that I is the maximal invariant set in U.

It is an easy consequence of the definition of an isolating block that the maximal invariant set contained in a block is isolated. Hence the block "isolates" a certain invariant set (which may sometimes be empty). Conversely we have the following theorem:

Theorem 2.7: If $I \subset M\text{-}C$ is a compact isolated invariant set of X then there exists an isolating block B such that I is the maximal invariant set contained in B.

For a proof see [4].

With this preparation we are now ready to say what it means to "regularize" the singularity C.

Definition 2.8: A closed subset C_1 which is relatively open in C is regularizable if there exists an isolating block $B \subset M\text{-}C$ such that for $x \in M\text{-}C$

(1) if (t_0, t_1) is the maximal interval such that $x \cdot (t_0, t_1)$ is defined and if $x \cdot t \to C_1$ as $t \to t_1$ then $x \cdot t$ must enter and stay in B as $t \to t_1$. Similarly if $x \cdot t \to C_1$ as $t \to t_0$ then $x \cdot t$ must enter and stay in B as $t \to t_0$.

(2) π: $b^+ - a^+ \to b^- - a^-$ admits a unique extension as a homeomorphism from b^+ to b^-.

Definition 2.9: Suppose that the singularity C is regularizable and B is as in 2.8. Then the regularized phase space for X is the space N obtained from M-C-int B by identifying points $x \in b^+$ with points $\pi(x) \in b^-$. More precisely define an equivalence relation $\sim$ on M-C-int B by $x \sim y$ if $x = y$ or if $x = \pi(y)$ or $y = \pi(x)$. Let N be the set of equivalence classes of points of M-C-int B and let ρ: M-C-int B $\to N$ be the natural projection. Give N the quotient topology. Then N is a manifold and ρ restricted to M-C-B is a homeomorphism. We identify M-C-B with ρ(M-C-B).

X induces a flow $\varphi: N \times R^1 \to N$ as follows:

(1) Suppose $p \cdot s \in$ M-C-B for each $s \in [0,t)$. Then define $\varphi(p,t) = \rho(p \cdot t)$.

(2) Suppose $p \in \rho(b)$ say $\rho^{-1}(p) = \{x, \pi(x)\}$. If $t \geq 0$ and $\pi(x) \cdot (0,t) \subset$ M-C-B define $\varphi(p,t) = \rho(\pi(x) \cdot t)$. If $t \leq 0$ and $x \cdot (t,0) \subset$ M-C-B define $\varphi(p,t) = \rho(x \cdot t)$.

(3) Extend $\varphi(p,t)$ by requiring that

$$\varphi(p, t_1 + t_2) = \varphi(\varphi(p, t_1), t_2).$$

Thus the flow is defined by following an integral curve of X until it hits b, crossing B in zero time and continuing along the appropriate integral curve of X.

The 2-body problem provides an example of the process of regularization by surgery. This example is discussed in [6].

3. The Planar 3-Body Problem

Three point masses move in the plane under the influence of their mutual gravitational attractions. We assume for simplicity that each particle has mass 1 and that the gravitational constant is equal to 1. Thus the state of the system is specified by a point $(Q,P) = (q_1,q_2,q_3,p_1,p_2,p_3) \in (R^2)^6$. Here q_j specifies the position of the jth particle and p_j specifies its momentum.

3.1: Define $C_{ij} = \{(Q,P): q_i = q_j\}$ for $i,j = 1,2,3$. Define $r_{ij} = |q_i - q_j|$ and $q_{ij} = q_i - q_j$. Let $C = C_{12} \cup C_{13} \cup C_{23}$. C is the set of "collision" states of the system.

The equations of motion can be formulated as a Hamiltonian system of differential equations defined on $R^{12}-C$.

3.2: Define $H: R^{12}-C \to R^1$ by

$$H(Q,P) = \frac{1}{2}(|p_1|^2 + |p_2|^2 + |p_3|^2) - (r_{12}^{-1} + r_{13}^{-1} + r_{23}^{-1})$$

H is the Hamiltonian function for the system and the equations of motion are

3.3:
$$\begin{aligned} \dot{q}_1 &= p_1 & \dot{p}_1 &= r_{12}^{-3}q_{21} + r_{13}^{-3}q_{31} \\ \dot{q}_2 &= p_2 & \dot{p}_2 &= r_{12}^{-3}q_{12} + r_{23}^{-3}q_{32} \\ \dot{q}_3 &= p_3 & \dot{p}_3 &= r_{13}^{-3}q_{13} + r_{23}^{-3}q_{23}. \end{aligned}$$

It is convenient to make the canonical change of variables

3.4: $$\xi = q_2 - q_3 \qquad \eta = \tfrac{1}{3}(-p_1 + 2p_2 - p_3)$$
$$x = q_1 - q_3 \qquad y = \tfrac{1}{3}(2p_1 - p_2 - p_3)$$
$$z = q_1 + q_2 + q_3 \qquad w = \tfrac{1}{3}(p_1 + p_2 + p_3)$$

Notice that w is one third the total momentum of the system and z specifies the center of mass of the system. Without loss of generality we assume that $w = z = 0$. In terms of the new variables the energy and angular momentum functions become

3.5: $$H(\xi, x, \eta, y) = H_1 + H_2 + H_3$$
$$= (|y|^2 - |x|^{-1}) + (|\eta|^2 - |\xi|^{-1}) + (y\cdot\eta - |x-\xi|^{-1}).$$

3.6: $$J(\xi, x, \eta, y) = \tfrac{2}{3}[(\xi) \times (\eta) + (x) \times (y)].$$

The equations of motion are

3.7: $$\dot{\xi} = 2\eta + y \qquad \dot{\eta} = -\xi|\xi|^{-3} - (\xi - x)|\xi - x|^{-3}$$
$$\dot{x} = 2y + \eta \qquad \dot{y} = -x|x|^{-3} - (x-\xi)|x-\xi|^{-3}$$

These equations are defined on $R^8 - C$ where $C = C_1 \cup C_2 \cup C_3 =$ $\{|\xi| = 0\} \cup \{|x| = 0\} \cup \{|x-\xi| = 0\}$.

The planar 3-body problem is the problem of studying the flow on R^8 with singularity C induced by equations 3.7. The equations 3.7 admit the functions H and J as integrals.

3.8: Define $M[h,\omega] = \{H=h\} \cap \{J=\omega\}$ for $h,\omega \in R^1$.

It is natural to study the flow on the integral surfaces $M[h,\omega]$ for various values of the parameters h and ω. A further reduction is possible by making use of the symmetry which exists in the problem. In what follows we treat ξ, x, η, y as complex variables (thus $\xi = \xi_1 + i\xi_2$, etc.).

3.9: Define an action $*$ of R^1 on R^8 by

$$z * t = e^{it}z \quad \text{where } z = (\xi, x, \eta, y) \in \mathbb{C}^4.$$

Both H and J are invariant with respect to this flow. In fact the flow $*$ is generated by the Hamiltonian function $\frac{3}{2}J$. Since the Poisson Bracket $[H, \frac{3}{2}J]$ is identically zero the 3-body flow generated by H and the flow $*$ commute. Notice also that $H(\bar{z}) = H(z)$ and $J(\bar{z}) = J(z)$ where $\bar{z}$ is the complex conjugate of z. We want to remove this "symmetry" from the problem.

3.10: Define $\bar{M}[h,\omega]$ to be the quotient space of $M[h,\omega]$ modulo the action $*$. Notice that $S = \{z \in M[h,\omega]: x|x|^{-1} = (1,0)\}$ is a global surface of section for the flow $*$ on R^8-C and every orbit of $*$ meets S exactly once. Hence $\bar{M}[h,\omega]$ is diffeomorphic to $S[h,\omega] = S \cap M[h,\omega]$. The 3-body flow on R^8-C induces a flow on S

in the following way: If we denote the 3-body flow on R^8-C by the notation $z \to z\cdot t$ then if $w \in S$ we define $w \cdot^* t = s$ where s is the intersection of the set $(w\cdot t) * R^1$ with S. The flow $\cdot^*$ obviously restricts to the surfaces $S[h,\omega]$.

As a first step in studying the 3-body flow it is useful to topologically characterize the surfaces $M[h,\omega]$ and $\widetilde{M}[h,\omega]$ which for most values of the parameters h and ω are manifolds. We include below a discussion of the topology of the surfaces $\widetilde{M}[h,\omega]$ which we will use later in section 5.

<u>3.11</u>: Define $\nabla = \{(s_1,s_2,s_3) \in S^2 : s_i > 0,\ s_i+s_j \le s_k\}$. ∇ is a spherical triangle with its corners removed. Notice that each point of ∇ specifies a unique triangle with sides s_1, s_2, s_3. We think of points of ∇ as specifying the "shape" of the triangle formed by the 3-bodies in the 3-body problem. The corners of ∇ correspond to the double collision states - namely those states where two particles occupy the same position. Recall that the positions of the three particles are specified by the vectors q_1, q_2, q_3 where in terms of the variables (ξ,x) we have $q_1 = \frac{2}{3}(x-\frac{1}{2}\xi)$, $q_2 = \frac{2}{3}(\xi-\frac{1}{2}x)$ and $q_3 = -\frac{1}{3}(\xi+x)$. Thus the triangle formed by the three particles has sides $|\xi|$, $|x|$ and $|x-\xi|$.

<u>3.12</u>: Define $\rho\colon R^8-C \to \nabla \times (0,\infty)$ by

$$\rho(\xi,x,\eta,y) = (r^{-1}|\xi|,\ r^{-1}|x|,\ r^{-1}|x-\xi|,\ r)$$

where

$$r = \sqrt{3}^{-1}(|q_1|^2 + |q_2|^2 + |q_3|^2)^{-\frac{1}{2}} = \frac{2}{9\sqrt{3}}(3|\xi|^2 + 3|x|^2 - 4\xi\cdot x)$$

3.13: Define $m[h,\omega] = \rho(M[h,\omega])$. $m[h,\omega]$ plays the role of a Hills region in our development. The following proposition characterizing $m[h,\omega]$ is proven in [7].

Proposition 3.14: $m[h,\omega] = \{(s,r) \in \nabla \times (0,\infty)\colon \frac{1}{3}r^{-2}(s_1^{-1} + s_2^{-1} + s_3^{-1})^2 + 2h\omega^2 \geq 0\}$.

$m[h,\omega]$ is shown in figure 2 below for three decreasing values of $h < 0$:

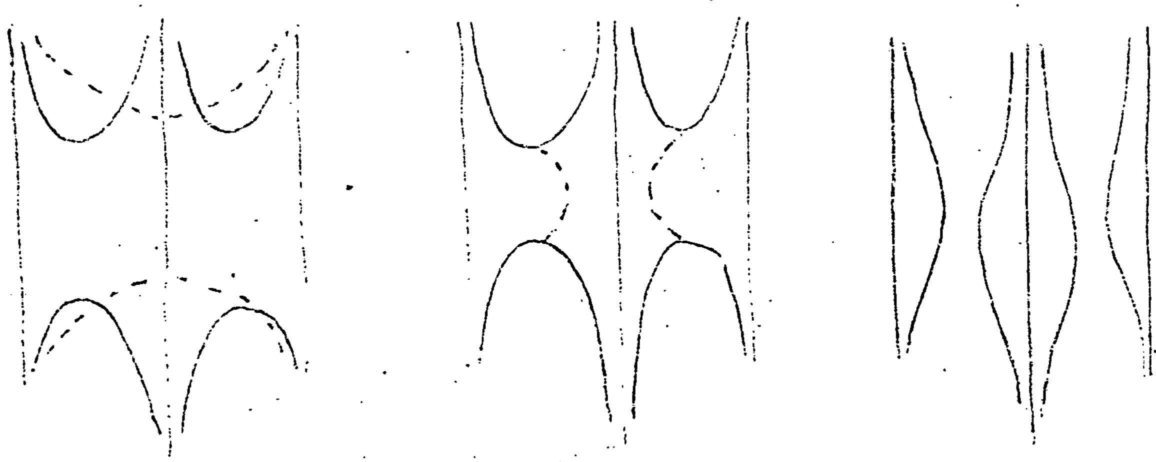

figure 2

3.15: Define $\ell[h,\omega]$ to be the quotient space $S^0 \times m[h,\omega]/\sim$ where $\sim$ is an equivalence relation defined by $1 \times (s,r) \sim -1 \times (s,r)$ whenever

$s \in \partial\nabla$. Thus $\ell[h,\omega]$ is the 3 manifold obtained by sewing two copies of $m[h,\omega]$ together along the set $m[h,\omega] \cap \partial\nabla \times R^1$.

The following proposition is essentially proven in [7].

Proposition 3.16: $S[h,\omega]$ is homeomorphic to the space obtained from $\ell[h,\omega] \times S^2$ by identifying $p \times S^2$ to a point whenever $p \in \partial\ell[h,\omega]$.

Sketch of the proof: Each point of $\ell[h,\omega]$ specifies the shape, size and orientation of the triangle formed by the three bodies in the plane. Thus each point of $\ell[h,\omega]$ corresponds to a unique point $(\xi,x) \in R^4$ such that $x|x|^{-1} = (1,0)$. Consider the set of (η,y) such that

$$\text{(a)} \quad |\eta|^2 + |y|^2 + y\cdot\eta = h + U(\xi,x)$$

$$\text{(b)} \quad (x) \times (y) + (\xi) \times (\eta) = \omega.$$

When (ξ,x) corresponds to a point belonging to the interior of $\ell[h,\omega]$ the set of (η,y) satisfying (a) and (b) is a 2-sphere. When (ξ,x) corresponds to a point belonging to the boundary of $\ell[h,\omega]$ the set of (η,y) satisfying (a) and (b) is a point and when (ξ,x) does not correspond to a point in $\ell[h,\omega]$ this set is empty. Thus $S[h,\omega]$ is a singular 2-sphere fibre bundle over $\ell[h,\omega]$. This fibre bundle turns out to be a product bundle.

4. Regularization of the 3-Body Problem

Recall that the equations of motion 3.8 define a flow with singularity C on R^8. In this section we construct three disjoint isolating blocks $B_i \subset R^8 - C$ such that the solutions which end in binary collisions must enter and remain in one of these blocks as the collision time is approached. We further show that the flow mappings across these blocks can be extended to diffeomorphisms of b_i^+ onto b_i^- for $i = 1,2,3$.

Definition 4.1: Let $\gamma(t)$ be a solution to the equations 3.7 and suppose that the maximum positive interval of time on which γ is defined is $[0,t_1)$. It is well known [9] that $\lim_{t \to t_1} \sigma(t)$ exists where $\sigma(t) = \frac{2}{3}(|x|^2 + |\xi|^2 - x \cdot \xi)$. $\sigma(t)$ is the moment of inertia of the system (in the old coordinates $\sigma = |q_1|^2 + |q_2|^2 + |q_3|^2$). If $0 < \sigma(t_1) < \infty$ then the solution $\gamma(t)$ is said to end in a binary collision. It is well known in this case that the limits $\lim_{t \to t_1} \xi(t)$, $\lim_{t \to t_1} x(t)$, $\lim_{t \to t_1} x(t) - \xi(t)$ exist and exactly one of these limits is zero.

In what follows we define for each $\epsilon > 0$ a set $B[\epsilon]$. We show that for sufficiently small $\epsilon > 0$ that this set is an isolating block for collisions of the type where $\xi(t) \to 0$ and we show that the flow map across $B[\epsilon]$ extends to a diffeomorphism of $b^+[\epsilon]$ onto $b^-[\epsilon]$. Let $B_1 = B[\epsilon]$. It is clear from the symetry of the problem that we may similarly construct isolating blocks B_2 and B_3 for collisions of the type where $x(t) \to 0$ and where $x(t) - \xi(t) \to 0$ respectively.

In order to define the isolating blocks $B[\epsilon]$ it is necessary to define some functions.

Definition 4.2:

(a) Choose a smooth function $\alpha\colon R^1 \to (0,1)$ having the properties that $\alpha(t) \leq -t^{-1}$ for $t \leq -1$.

(b) Choose a smooth function $\beta\colon (0,\infty) \to (0,1)$ having the properties:

(1) $t^{-1}\beta(t) \leq 1$

(2) $\beta'(t) \leq 1$ and $\beta'(t) = 0$ if $t \geq 1$

(3) $\beta''(t) \leq 1$

(c) Define for $\epsilon > 0$ $F_\epsilon: R^8-C \to R^1$ by $F_\epsilon(\xi,x,\eta,y) = |\xi|^2 - \epsilon\alpha(H(\xi,x,\eta,y))\beta(\ell^2)$ where $\ell = |2x-\xi|$.

(d) Define for $\epsilon > 0$ $G_\epsilon: R^8 \to R^1$ by $G_\epsilon(\xi,x,\eta,y) = |\eta|^2 - \epsilon|y|^2$

<u>4.3</u>: Define $B[\epsilon] = \{(\xi,x,\eta,y) \in R^8-C\colon F_\epsilon \leq 0,\ G_\epsilon \leq 0\}$.

It will be shown that for ϵ sufficiently small $B[\epsilon]$ is an isolating block for the flow on R^8-C. Our choice of B is motivated by the following considerations. Suppose $\gamma(t)$ is an orbit which ends in a binary collision of mass 2 with mass 3 at time t_1. (i.e., $|\xi(t)| \to 0$ as $t \to t_1$). Then it is known [9] that $x(t)$ and $x(t)-\xi(t)$ approach finite limits as $t \to t_1$. Hence there exists $\tau < t_1$ such that if $\tau \leq t < t_1$ then $F_\epsilon(\gamma(t)) \leq 0$. Furthermore it is known that $|\eta(t)|$ approaches infinity as $t \to t_1$ while $y(t)$ approaches a finite limit as $t \to t_1$. Hence there exists $\tau < t_1$ such that $\tau \leq t < t_1$ implies $G_\epsilon(\gamma(t)) \leq 0$. Therefore we have established the following proposition.

<u>Proposition 4.4</u>: For any $\epsilon > 0$, if $\gamma(t)$ is a solution of the equations 3.7 whose maximal positive interval of existence is $[0,t_1)$ and

$\gamma(t)$ ends in a binary collision where $\lim_{t \to t_1} \xi(t) = 0$, then there exists $0 \le \tau < t_1$ such that $\gamma(t) \in B[\epsilon]$ for all $t \in [\tau, t_1)$.

Our next objective is to show that $B[\epsilon]$ is an isolating block for sufficient small ϵ.

Proposition 4.5: Given $\delta > 0$ there exists $\epsilon > 0$ such that for any point $z = (\xi, x, \eta, y)$ belonging to $B[\epsilon]$,

(a) $\dot{F}(z) = 0$ implies $\xi \cdot \eta \le \delta |\xi||\eta|$

(b) $\dot{G}(z) = 0$ implies $\xi \cdot \eta \le \delta |\xi||\eta|$

(c) $\ddot{F}(z) \ge |\xi|^{-1}$

(d) $\dot{G}(z) = 0$ implies $\ddot{G}(z) > 0$

Proof of (a): $\dot{F} = 4\xi \cdot \eta - R$ where

$$R = \epsilon\alpha\beta\,[6x \cdot y - 3\xi \cdot y] - 2(\xi \cdot y)$$

$\dot{F} = 0$ implies that $\xi \cdot \eta = [\frac{1}{4} R\, |\xi|^{-1} |\eta|^{-1}] |\xi||\eta|$. Since $F_\epsilon \le 0$ and $G_\epsilon \le 0$ we compute that $|R| \le 9\epsilon^3 |\xi||\eta| + 2\epsilon|\xi||\eta|$ and hence $|\frac{1}{4} R |\xi|^{-1} |\eta|^{-1}| \le \frac{1}{4}(9\epsilon^3 + 2\epsilon)$. By choosing ϵ sufficiently small we can make this expression less than δ.

Proof of (b):

$$\dot{G} = 2\epsilon[(\xi \cdot \eta)|\xi|^{-3} + (\xi - x) \cdot \eta |\xi - x|^{-3}] - 2[(x \cdot y)|x|^{-3} + (x - \xi) \cdot y |x - \xi|^{-3}].$$

$\dot{G} = 0$ if and only if $\xi \cdot \eta - \frac{1}{\epsilon} R = 0$ where

$$R = [2\epsilon(x-\xi)\cdot\eta|\xi-x|^{-3} + 2(x\cdot y)|x|^{-3} + (x-\xi)\cdot y|x-\xi|^{-3}]|\xi|^3$$

Choose ϵ so small that $|x-\xi|^{-3} \leq 2|x|^{-3}$. Notice that by the choice of β, $|\xi|^2|x|^{-2} \leq \epsilon$. Hence

$$|R| \leq |\xi||\eta|[4\epsilon|x|^{-2}|\xi|^2 + 2\epsilon|x|^{-2}|\xi|^2 + 2|x|^{-2}|\xi|^2\epsilon]$$

$$\leq \epsilon|\xi||\eta|[8|x|^{-2}|\xi|^2].$$

Therefore $\xi\cdot\eta \leq (\frac{1}{\epsilon}|R||\xi|^{-1}|\eta|^{-1})|\xi||\eta| \leq (8\epsilon)|\xi||\eta|$.

<u>Proof of (c)</u>: $\ddot{F} = 4[\dot{\xi}\cdot\eta + \xi\cdot\dot{\eta}] - \dot{R}$. Given $\delta > 0$ there exists $\epsilon > 0$ such that $\dot{\xi}\cdot\eta + \xi\cdot\dot{\eta} \geq (1-\delta)|\xi|^{-1}$ on $B[\epsilon]$. To see this we compute that

$$\dot{\xi}\cdot\eta + \xi\cdot\dot{\eta} = 2|\eta|^2 - |\xi|^{-1} + (y\cdot\eta) - \xi\cdot(\xi-x)|\xi-x|^{-3}.$$

By choosing ϵ small we can obtain the inequality

$$\dot{\xi}\cdot\eta + \xi\cdot\dot{\eta} \geq (1-\delta_1)[2|\eta|^2 - |\xi|^{-1}].$$

where $\delta_1 = \delta_1(\epsilon) \to 0$ as $\epsilon \to 0$.

Now

$$|\eta|^2 = h + |\xi|^{-1} + |x|^{-1} + |x-\xi|^{-1} - |y|^2 - y\cdot\eta \geq (1-\delta_2)[h+|\xi|^{-1}].$$

where $\delta_2 = \delta_2(t) \to 0$ as $\epsilon \to 0$. Hence $2|\eta|^2 - |\xi|^{-1} \geq (2h + |\xi|^{-1})(1-\delta_2)$. However, by choice of α, $2h + |\xi|^{-1} \geq (1-\delta_3)|\xi|^{-1}$ where $\delta_3 = \delta_3(\epsilon) \to 0$ as $\epsilon \to 0$. Thus it remains to show that $|\dot{R}| < (4-\delta)|\xi|^{-1}$ on $B[\epsilon]$ for sufficiently small ϵ.

$$\dot{R} = \epsilon\alpha\beta'[9|y|^2+(6x-3\xi)\cdot\dot{y}] + \epsilon\alpha\beta'[6(x\cdot y)-3(\xi\cdot y)]^2 - 2[2|y|^2+4(\eta\cdot y)+\xi\cdot\dot{y}].$$

For sufficiently small ϵ the following estimates hold on $B[\epsilon]$:

$$|\dot{y}| \leq 3|x|^{-2}, \quad |\xi||\eta|^2 \leq 2, \quad |\xi|^2|x|^{-2} \leq \epsilon.$$

Using these estimates it is easy to show that $|\xi||\dot{R}| \to 0$ as $\epsilon \to 0$. Hence given $\delta > 0$ there exists $\epsilon > 0$ such that $|\dot{R}| < \delta|\xi|^{-1}$ on $B[\epsilon]$. It follows that $\ddot{F} \geq (4-2\delta)|\xi|^{-1}$ on $B[\epsilon]$ for sufficiently small $\epsilon > 0$.

<u>Proof of (d)</u>: $\ddot{G} = 2|\dot{y}|^2 + 2y\cdot\ddot{y} - 2\epsilon(|\dot{\eta}|^2 + \eta\cdot\ddot{\eta})$ and

$$\ddot{y} = -(2y+\eta)|x|^{-3} - (y-\eta)|x-\xi|^{-3} + 3x|x|^{-5}[x\cdot(2y+\eta)]$$
$$+ 3(x-\xi)|x-\xi|^{-5}(x-\xi)\cdot(y-\eta).$$

We estimate $y\cdot\ddot{y} \geq -5\epsilon|\eta|^2|x|^{-3}$ on $B[\epsilon]$. Also $\eta\cdot\ddot{\eta} = -3|\eta|^2|\xi|^{-3} + 3(\xi\cdot\eta)(\xi\cdot y)|\xi|^{-5} + (\eta\cdot y)|\xi|^{-3} + \mathscr{E}$ where $|\mathscr{E}| \leq \delta(\epsilon)|\eta\cdot\ddot{\eta}|$ and $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. Fix $0 < \delta < 1$. Then $\dot{G} = 0$ implies $|\xi\cdot\eta| \leq \delta|\xi||\eta|$ for sufficiently small ϵ by (b). We estimate that

$$\eta\cdot\ddot{\eta} \leq -3|\eta|^2|\xi|^{-3} + \epsilon^{1/2}|\eta|^2|\xi|^{-3} + 3\delta|\eta|^2\epsilon|\xi|^{-3} \leq -(3-\delta_1)|\xi|^{-3}|\eta|^2$$

where $\delta_1 = \epsilon^{1/2} + 3\delta\epsilon$. Since given $\delta_2 > 0$, $|\dot{\eta}|^2 = (1+\delta_2)|\xi|^{-4}$ for ϵ sufficiently small we obtain for small ϵ the inequality

$$-(|\dot{\eta}|^2 + \eta\cdot\ddot{\eta}) \geq (2-\delta)|\xi|^{-4}.$$

Hence for small ϵ,

$$\begin{aligned}\ddot{G} &\geq -5\epsilon|\eta|^2|x|^{-3} + (2\epsilon)(2-\delta)|\eta|^2|\xi|^{-3}\\ &\geq \epsilon|\eta|^2|\xi|^{-3}[-5|\xi|^3|x|^{-3} + 2(2-\delta)]\\ &\geq \epsilon|\eta|^2|\xi|^{-3}[4-\delta-\epsilon^{3/2}] > 0.\end{aligned}$$

<u>Proposition 4.7</u>: $B[\epsilon]$ is diffeomorphic to $(D^2-0)\times S^1\times D^2\times S^1\times R^1\times R^1$.

<u>Proof</u>: Define $\varphi\colon (D^2-0)\times S^1\times D^2\times S^1\times R^1\times R^1 \to B[\epsilon]$ as follows:

$$\varphi(d_1,s_1,d_2,s_2,\ell,h) = (\xi,x,\eta,y) \quad \text{where}$$

$$\xi = \epsilon\alpha(h)\beta(\ell)d_1 \quad \text{and}$$

$$x = \lambda s_2 \quad \text{where } \lambda > 0 \text{ satisfies the equation}$$

$$\ell^2 = 4\lambda^2 + |\xi|^2 - 4\lambda\xi\circ s_2.$$

Then $\ell = |2x-\xi|$, and $F(\xi,x) \leq 0$.

Observe that $\{(a,b) \in S^3: |a|^2 \leq \epsilon|b|^2\}$ is diffeomorphic to $D^2 \times S^1$. We map $(d_2, s_1) \in D^2 \times S^1$ to $(a,b) \in S^3$ where $a = (\frac{\epsilon}{1+\epsilon})^{1/2} d_2$ and $b = s_1(1-|a|^2)^{1/2}$. Define $\mu > 0$ by the equation $\mu^2 T(a,b) = h+U(\xi, x)$. Finally define

$$\eta = \mu\epsilon^{1/2}(1+\epsilon)^{1/2} d_2 \quad \text{and} \quad y = \mu[1-\epsilon(1+\epsilon)^{-1}|d_2|^2]^{1/2} s_1.$$

Notice that η and y satisfy the relation $|y|^2 \leq \epsilon|\eta|^2$ and that $H(\xi, x, \eta, y) = h$. φ is the desired diffeomorphism.

<u>Corollary 4.8:</u> $b[\epsilon]$ is diffeomorphic to

$$S^1 \times S^1 \times [(0,1] \times \partial D^2 \cup 1 \times D^2] \times S^1 \times R^1 \times R^1$$

$$= S^1 \times S^1 \times R^2 \times S^1 \times R^1 \times R^1.$$

It is important to notice that $b[\epsilon] \cap \{\xi, x, \eta, y: \xi \cdot \eta \leq 0\}$ is diffeomorphic to $A^+ \times R^2 \times S^1 \times R^1 \times R^1$ where

$$A^+ = \{(s_1, s_2) \in S^1 \times S^1: s_1 \cdot s_2 \leq 0\}.$$

The 3-body flow on R^8-C defines a map π across the block B (see 2.4). Our next goal is to show that π admits a unique extension as a diffeomorphism of b^+ onto b^-. We introduce new coordinates by a Levi-Civita transformation and perform an isoenergetic reduction to obtain a new vector field which is well behaved. We use this vector field to show that the map π extends to all of b^+.

4.9: Define $\mathscr{T}: \mathbb{C} - 0 \times \mathbb{C}^3 \to \mathbb{C}^4$ by

$$\mathscr{T}(u,x,v,y) = (\tfrac{1}{2} u^2, x, v(\bar{u})^{-1}, y).$$

$\mathscr{T}$ is a canonical transformation of the type used by Levi-Civita.

4.10: Fix $h \in R^1$ and define $G: \mathbb{C}^4 \to R^1$ by

$$G(u,x,v,y) = |u|^2[H \circ \mathscr{T}(u,x,v,y) - h].$$

Then $G(u,x,v,y) = A|u|^2 + y \cdot uv + |v|^2 - 1$ where $A = |y|^2 - |x|^{-1} - |x - \frac{1}{2}u^2|^{-1} - h$. Define the vector field X_G on $\mathbb{C}^4$ by

$$\dot{u} = \frac{\partial G}{\partial v}, \quad \dot{v} = -\frac{\partial}{\partial u}G, \quad \dot{x} = \frac{\partial}{\partial y}G, \quad \dot{y} = -\frac{\partial}{\partial x}G.$$

It is well known that $\mathscr{T}$ takes the surface $\{G = 0\}$ to the surface $\{H = h\}$ and that on $\{G = 0\}$, $D\mathscr{T}(X_G) = |u|^2 X_H$ where X_H is the 3-body vector field.

The vector field X_G is given explicitly by the equations

4.11:

$$\begin{aligned}
\dot{u} &= 2v + yu \\
\dot{v} &= -2Au - yv - (x - \tfrac{1}{2}u^2)|x - \tfrac{1}{2}u^2|^{-3}|u|^2 \\
\dot{x} &= 2y|u|^2 + uv \\
\dot{y} &= -x|x|^{-3}|u|^2 - [x|x|^{-3} + (\tfrac{1}{2}u^2 - x)|x - \tfrac{1}{2}u^2|^{-3}]|u|^2
\end{aligned}$$

Notice that X_G is defined when $u = 0$.

Definition 4.12: $\widetilde{B}[h] = \mathscr{T}^{-1}(B[h])$ where $B[h] = B \cap \{H = h\}$. Then

$$\widetilde{B}[h] = \left\{ (u,x,v,y): \begin{array}{l} |u|^4 \leq \epsilon\alpha(h)\beta(\ell) \\ |y|^2|u|^2 \leq \epsilon|v|^2 \\ A|u|^2 + y\cdot uv + |v|^2 = 1 \end{array} \right\} .$$

Notice that for $\epsilon > 0$ sufficiently small, $\widetilde{B}[h]$ is an isolating block for the vector field X_G on the surface $\{G = 0\}$. Let $\widetilde{\pi}$ be the map across $\widetilde{B}[h]$ defined by the flow generated by X_G.

Proposition 4.13: $\widetilde{\pi}$ is a diffeomorphism of $\widetilde{b}^+$ onto $\widetilde{b}^-$.

Lemma 4.14: Given $\delta > 0$ one can choose $\widetilde{B}[h]$ sufficiently small (i.e. choose ϵ sufficiently small in the definition of $\widetilde{B}[h]$) so that $1-\delta \leq |v| \leq 1+\delta$ whenever $(u,x,v,y) \in \widetilde{B}[h]$.

Proof: Since $|y|^2|u|^2 \leq \epsilon|v|^2$ on $\widetilde{B}[h]$, we have $|y\cdot uv| \leq \epsilon^{1/2}|v|^2$. We estimate that $|A||u|^2 \leq \epsilon|v|^2 + 3|x|^{-1}|u|^2 + |u|^2|h|$. Since $|u|^4 \leq \epsilon\alpha(h)\beta(\ell)$ it follows that $|x|^{-1}|u|^2 \leq \epsilon^{1/2}$ and hence $|A||u|^2 \leq \epsilon|v|^2 + 3\epsilon^{1/2} + \epsilon^{1/2}|h|$. Now since $\widetilde{B}[h] \subset \{G = 0\}$ we must have $A|u|^2 + y\cdot uv + |v|^2 = 1$. We have shown that $|A|u|^2 + y\cdot uv| \leq (\epsilon+\epsilon^{1/2})|v|^2 + \epsilon^{1/2}(3+|h|)$. It follows that for ϵ sufficiently small, $|1 - |v|^2| \leq \delta$ which is the desired result.

Lemma 4.15: Suppose that $\gamma\colon [0,T] \to \widetilde{B}[h]$ is an integral curve of the vector field X_G with $\gamma(t) = (u(t),x(t),v(t),y(t))$. Then $T \leq 2|u(0)|$.

Proof: For $t \in [0,\tau]$ where $\tau = 2|u_0|$ the following estimates can be obtained using equations 4.11 and lemma 4.14 and the definition of B:

$$|\dot{u}(t)| \leq 3, \quad 0 \leq |u(t)| \leq 7|u_0| \quad \text{(this uses 4.14)},$$

$$|x(t)-x_0| \leq \delta_1(\epsilon)|x_0|, \quad |y(t)-y_0| \leq \delta_2(\epsilon)|y_0|,$$

$$|v(t)-v_0| \leq \delta_3(\epsilon)|v_0|,$$

where $\delta_i(t) \to 0$ as $\epsilon \to 0$ for $i = 1,2,3$. Since $\dot{u} = 2v + yu$ we have $|u(\tau)| \geq \frac{3}{2}|u_0|$. However $\ell(\tau) - \ell(0) \leq 20|u_0|^2$ and this implies that $|u(\tau)|^4 > \epsilon\alpha(h)\beta(\ell(\tau))$ and hence that $\gamma(\tau) \notin B$. This is a contradiction showing that $\gamma(t)$ cannot remain in B for $t \geq 2|u_0|$. This completes the proof.

Proof of Proposition 4.13: It is sufficient to show that if $\gamma(t)$ is an integral curve of X_G with $\gamma(0) \in \tilde{b}^+$, then $\gamma(t)$ crosses $\tilde{B}[h]$ in a finite time. Observe that $\dot{u}$, $\dot{x}$ and $\dot{y}$ are bounded on $\tilde{B}[h]$. Hence any integral curve $\gamma(t)$ of X_G must be bounded over a finite time interval. Since X_G is without singularity in $\tilde{B}[h]$, $\gamma(t)$ must either cross $\tilde{B}[h]$ in finite time or must be defined over an infinite time interval and remain in $\tilde{B}[h]$. However lemma 4.15 rules out this second possibility. This completes the proof.

Corollary 4.16: π admits a unique extension as a diffeomorphism from b^+ to b^-.

Proof: Define $\hat{\pi}$: $b^+[h] \to b^-[h]$ by $\hat{\pi} = T\tilde{\pi}T^{-1}$. Since $(D\mathscr{T})X_G =$

$|u|^2 X_H$ it follows that $\hat{\pi} = \pi$ on $b^+[h] - a^+[h]$. Hence $\hat{\pi}$ is a continuous extension of π to $b^+[h]$. Since h was arbitrary, π admits a continuous extension to all of b^+. Let $\tilde{a}^+$ be the set of points p in $\tilde{b}^+$ such that the integral curve of X_G through p meets the set $\{(u,x,v,y): u = 0\}$ as it crosses $\tilde{B}$. Any point in $\tilde{a}^+$ is the limit of points in $\tilde{b}-\tilde{a}^+$ and it follows that any point in a^+ is the limit of points in $b^+ - a^+$. Since $\hat{\pi} = \pi$ on $b^+ - a^+$ the extension $\hat{\pi}$ must be unique.

In section 5 we will need to know more about the map π: $b^+ \to b^-$. The following lemma provides the desired information.

Lemma 4.17: Given $\delta > 0$ there exists $\epsilon > 0$ such that if $(\xi,x,\eta,y) \in b^+[\epsilon]$, if angles θ,φ are defined by $\xi = |\xi|(\sin\theta, \cos\theta)$ and $\eta = |\eta|(\sin\varphi, \cos\varphi)$, and if $\pi(\xi,x,\eta,y) = (\xi_1,x_1,\eta_1,y_1)$ then

(1) $|x-x_1| < \delta|x|$, (2) $||\xi|-|\xi_1|| < \delta$,

(3) $|y-y_1| < \delta|y|$, (4) $||\eta|-|\eta_1|| < \delta|\eta|$,

(5) $|\theta_1-(\theta+4(\varphi-\theta))| < \delta$ where $\xi_1 = |\xi_1|(\sin\theta_1, \cos\theta_1)$

(6) $|\varphi_1-(3\varphi-2\theta-\pi)| < \delta$ where $\eta_1 = |\eta_1|(\sin\varphi_1, \cos\varphi_1)$.

Proof: It is shown in the proof of 4.15 that (1), (3), and (4) hold. Making use of the fact that $H(\xi,x,\eta,y) = H(\xi_1,x_1,\eta_1,y_1)$ one can show that (2) follows from (1), (3) and (4). Also in 4.15 it is shown that $|v_1-v| \leq \delta|v|$ for ϵ sufficiently small where $\mathcal{T}(u,v,x,y) = (\xi,x,\eta,y)$ and $\tilde{\pi}(u,v,x,y) = (u_1,v_1,x_1,y_1)$. Using this fact and the definition of $\mathcal{T}$ conditions (5) and (6) can be shown to follow.

5. The Topology of the Integral Surfaces

In section 3 we described $S[h,\omega]$ as a singular 2-sphere fibre bundle over a "Hills region" $\ell[h,\omega]$. Our goal in this section is to describe the topology of the regularized phase space $R[h,\omega]$ for the 3-body flow on $S[h,\omega]$. In figure 3 below we show $\ell[h,\omega]$ for $\omega \neq 0$ and $h \ll 0$. In this case $\ell[h,\omega]$ has three components.

5.1: Define $B[h,\omega,\epsilon] = B[\epsilon] \cap S[h,\omega]$.

The shaded region represents the projection of the isolating block $B[h,\omega,\epsilon]$ into $\ell[h,\omega]$ and the three dotted rays correspond to the binary collision sets C_1, C_2 and C_3.

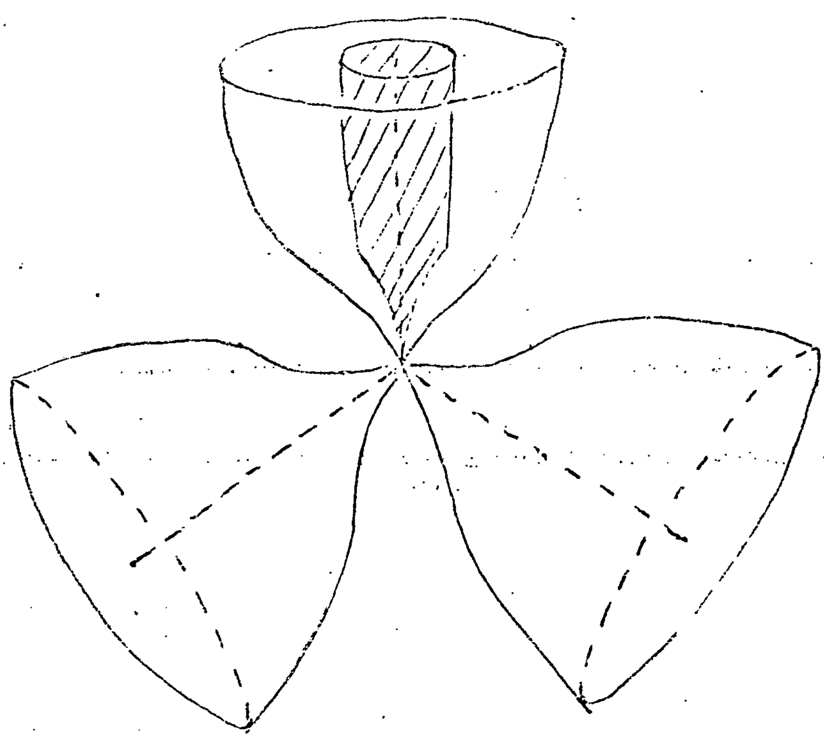

figure 3

We choose ϵ sufficiently small so that φ given in proposition 4.6 is a homeomorphism and $B[\epsilon]$ is a regularizing block for the singularity C_1.

5.2: Define $B_1 = B[h,\omega,\frac{1}{2}\epsilon]$. Then B_1 is an isolating block for the 3-body flow on $S[h,\omega]$.

5.3: Define $W_1 = \{(\xi,x,\eta,y) \in S[h,\omega]$: for some (η_1,y_1) $(\xi,x,\eta_1,y_1) \in B[h,\omega,\epsilon]\}$.

5.4: Define an equivalence relation $\sim$ on W_1 - int B_1 by setting $z \sim z'$ if $z = z'$ or if $z \in b_1^+$ and $z' = \pi(z)$ or if $z' \in b_1^+$ and $z = \pi(z')$. Define $\hat{W}_1$ to be the quotient space $(W_1\text{-int } B_1)/\sim$.

It is clear that we may similarly define W_2, W_3, $B_2 \subset W_2$, $B_3 < W_3$, $\hat{W}_2$ and $\hat{W}_3$. B_2 and B_3 are isolating blocks for the flow on $S[h,\omega]$ with the property that any solution in $S[h,\omega]$ which tends to the binary collision set C_j must enter B_j for $j = 2,3$.

5.5: Define $R[h,\omega]$ to be the regularized phase space for the 3-body flow on $S[h,\omega]$. Then

$$R[h,\omega] = \{S[h,\omega] - \bigcup_{j=1}^{3} \text{int } W_j\} \cup \{\bigcup_{j=1}^{3} \hat{W}_j\}$$

where the boundary of $\hat{W}_j$ is identified with the boundary of W_j.

We can characterize $R[h,\omega]$ once we know the topology of the sets $\hat{W}_j$ and $\{S[h,\omega] - \bigcup_{j=1}^{3} \text{int } W_j\}$. The following propositions characterize the topology of the sets W_j and $\hat{W}_j$.

Proposition 5.6: Let $W = W_1$. For $\epsilon > 0$ sufficiently small the projection of W into $\mathbb{C}^2$ is homeomorphic to $R^1 \times (D^2-0)$ and W is homeomorphic to $R^1 \times (D^2-0) \times S^2$.

Proof: Define $P\colon S[h,\omega] \to \mathbb{C}^2$ by $P(\xi,x,\eta,y) = (\xi,x)$. Fix $x = (|x|,0)$ and fix $\hat{\xi}$ a unit vector. Define $\lambda(x,\hat{\xi},\epsilon)$ to be the maximum positive number such that $(x,\lambda(x,\hat{\xi},\epsilon)\hat{\xi}) \in P(W)$. Then

$$\lambda(x,\hat{\xi},\epsilon) = \sup\{\lambda\colon \text{(a), (b), (c), (d) below have a solution } \eta_1,y_1\}.$$

(a) $\lambda^2 \leq \epsilon\alpha(h)\beta(|2x-\lambda\hat{\xi}|^2)$

(b) $|y|^2 \leq \epsilon|\eta|^2$

(c) $(x)\times(y) + (\lambda\hat{\xi})\times(\eta) = \omega$

(d) $|\eta|^2 + |y|^2 + y\cdot\eta = h + U(\lambda\hat{\xi},x)$.

Define $\lambda_a(x,\hat{\xi},\epsilon) = \max\{\lambda\colon \lambda$ is a solution of (a)$\}$ and define $\lambda_1(x,\hat{\xi},\epsilon) = \max\{\lambda\colon \lambda \leq \lambda_a(x,\hat{\xi},\epsilon),\ \lambda \leq 2\epsilon|x|^2|\omega|^{-1}\}$.

Lemma 5.7: If $0 < \lambda \leq \lambda_1(x,\hat{\xi},\epsilon)$ then $S^2(\lambda) = \{(\eta,y)\colon (\eta,y)$ satisfy (c) and (d)$\}$ is a 2-sphere.

Proof: It is sufficient to show that the plane $C(\lambda) = \{(\eta,y)$ which satisfy (c)$\}$ contains a point inside the sphere $D(\lambda) = \{(\eta,y)$ which satisfy (d)$\}$. $D(\lambda)$ contains the ball of radius R where $R^2 = \frac{1}{2}[h+U(\lambda\hat{\xi},x)]$. The plane $C(\lambda)$ is distance ρ from zero where $\rho^2 =$

$\omega^2(|x|^2+\lambda^2)^{-1}$. Thus it is sufficient to show that $\rho^2 < R^2$. For ϵ sufficiently small and $0 < \lambda < \lambda_1(x,\hat{\xi},\epsilon)$ we estimate that $R^2 \geq \frac{1}{3}\lambda^{-1}$. The inequality $\rho^2 < R^2$ follows from the inequality $\lambda < \frac{1}{6}|x|^2\omega^{-2}$ which holds whenever $2\epsilon < \frac{1}{6}$. This completes the proof of the lemma. The following lemma is also needed:

Lemma 5.8: $\lambda(x,\hat{\xi},\epsilon)$ is a continuous function of x and $\hat{\xi}$ and $(x,\xi) \in P(W)$ if and only if $0 < |\xi| \leq \lambda(x,\xi|\xi|^{-1},\epsilon)$. Furthermore for $\omega = 0$, $\lambda(x,\hat{\xi},\epsilon) = \lambda_a(x,\hat{\xi},\epsilon)$ and for $\omega \neq 0$, $\lambda(x,\hat{\xi},\epsilon) \leq \lambda_1(x,\hat{\xi},\epsilon)$.

Proof: Consider the equations

$$(c)' \quad (x) \times (y) = \omega \qquad\qquad (d)' \quad |\eta|^2 = \lambda^{-1/2}$$

Let $\lambda_2(x,\hat{\xi},\epsilon) = \max\{\lambda: (a), (b), (c)', (d)' \text{ have a solution}\}$. For ϵ sufficiently small the equations (c)' and (d)' approximate the equations (c) and (d) and we must have $\lambda(x,\hat{\xi},\epsilon) \leq 2\lambda_2(x,\hat{\xi},\epsilon)$. We compute that $\lambda_2(x,\hat{\xi},\epsilon) = \max\{\lambda: \lambda \leq \lambda_a(x,\hat{\xi},\epsilon),\ \lambda \leq \epsilon|x|^2|\omega|^{-1}\}$. Notice that for $\omega = 0$, (b), (c), (d) always have a solution and hence for $\omega = 0$, $\lambda(x,\hat{\xi},\epsilon) = \lambda_a(x,\hat{\xi},\epsilon)$.

Now suppose $\omega \neq 0$. The 2-sphere $S^2(\lambda)$ varies continuously with λ. Thus if $\lambda = \lambda(x,\hat{\xi},\epsilon)$, $S^2(\lambda)$ must intersect only the boundary of the cone $K = \{(\eta,y) \text{ which satisfy (b)}\}$. In this case the equations (b), (c), (d) can be written as one equation

(e) $$\hat{E}(\lambda, x, \hat{\xi}) \geq 0 \quad \text{where}$$

$$E(\lambda, x, \hat{\xi}) = \sup\{e(\lambda, x, \hat{\xi}, \hat{\eta}, \hat{y}) : \hat{\eta}, \hat{y} \in S^1\} \quad \text{and}$$

$$e(\lambda, x, \hat{\xi}, \hat{\eta}, \hat{y}) = [h+U]^{1/2}((x)\times(\hat{y})\epsilon^{1/2} + (\lambda\hat{\xi})\times(\hat{\eta}))$$
$$- \omega[1+\epsilon+\epsilon^{1/2}(\hat{\eta}\cdot\hat{y})].$$

Notice that $e(\lambda, x, \hat{\xi}, \hat{\eta}, \hat{y})$ is continuous in all its variables and that for $0 < \lambda \leq \lambda_2(x, \hat{\xi}, \epsilon)$, we have $\frac{\partial}{\partial\lambda}e < 0$. It follows that E is continuous and that E is monotonically decreasing in λ for fixed $(x, \hat{\xi})$. Therefore $\lambda(x, \hat{\xi}, \epsilon) = \max\{\lambda : E(\lambda, x, \hat{\xi}) \geq 0\}$ is also continuous and $(x, \xi) \in P(W)$ if and only if $0 < \lambda \leq \lambda(x, \hat{\xi}, \epsilon)$.

We now give the proof of proposition 5.6. Define $f: R^1 \times (D^2-0) \to P(W)$ by $f(t,d) = (\xi, x)$ where $x = t(1,0)$ and $\xi = \lambda(x, d|d|^{-1}, \epsilon)d$. f is the desired homeomorphism. For $(\xi, x) \in P(W)$ $\{(\eta, y) : (\xi, x, \eta, y) \in W\}$ is a 2-sphere by the previous lemmas. Thus W is a 2-sphere fibre bundle over $P(W)$ and the 2-sphere fibres do not "twist". Hence W is homeomorphic to $R^1 \times (D^2-0) \times S^2$.

<u>Definition 5.7</u>: Let $X = \{x \in R^3 : x_1^2+x_2^2 \leq 1,\ x_1^2+x_2^2+x_3^2 \geq 1/2,\ -2 < x_3 < 2\}$ Let $X_1 = \{x \in X : -1 \leq x_3 \leq 1\}$. Let $Y = \{x \in X : x_1^2+x_2^2 = 1\}$ and let (θ, φ, x_3) be coordinates on $S^1 \times Y$ where θ and φ are angular variables. Define

$$\mathscr{A}^+ = \{(\theta,\varphi,x_3) \in S^1 \times Y\colon \theta + \pi/2 \leq \varphi \leq \theta - \pi/2\}$$

$$\mathscr{A}^- = \{(\theta,\varphi,x_3) \in S \times Y\colon \theta - \pi/2 \leq \varphi \leq \theta + \pi/2\}$$

$f\colon \mathscr{A}^+ \to \mathscr{A}^-$ by

$$f(\theta,\varphi,x_3) = (\theta + 4(\varphi-\theta), 3\varphi-2\theta-\pi, x_3).$$

Define $\{R^1 \times P^3 - S^1 \times D^3\}$ to be the space obtained from $S^1 \times X$ by identifying points of $S^1 \times Y$ which correspond to each other by the map f. The notation is meant to be suggestive since this space is homeomorphic to $R^1 \times P^3$ (where P^3 denotes real projective space) minus a set homeomorphic to $S^1 \times D^3$.

<u>Proposition 5.8</u>: Let $W = W_1$ and let $B = B_1$. For ϵ sufficiently small there exists a homeomorphism $\sigma\colon W - \text{int } B \to R^1 \times S^1 \times X$ and a homeomorphism $\gamma\colon R^1 \times S^1 \times X \to R^1 \times S^1 \times X$ such that the diagram

$$\begin{tikzcd} \mathscr{A}^+ \arrow[r, "f"] \arrow[d, "\gamma"'] & \mathscr{A}^- \arrow[d, "\gamma"] \\ \sigma(b^+) \arrow[r, "\sigma\pi\sigma^{-1}"] & \sigma(b^-) \end{tikzcd}$$

commutes. Notice that $W - \text{int } B$ depends on ϵ.

<u>Corollary 5.9</u>: $\hat{W}_1 = \hat{W}$ is homeomorphic to $(R^1 \times P^3 - S^1 \times D^3)$.

<u>Proof</u>: The homeomorphism is $\gamma\circ\sigma$.

Before giving the proof of 5.8 we include some motivation.

Fix $\hat{\xi}$ a unit vector and fix $x = (|x|,0)$. Then the part of W over the ray $\{(\lambda\hat{\xi},x): 0 < \lambda \leq \lambda(x,\hat{\xi},\epsilon)\}$ is a 2-sphere fibre bundle as shown in figure 4 below. B intersects each fibre for $0 < \lambda \leq \lambda(x,\hat{\xi},\epsilon/2)$ in a shaded annulus as shown. Thus the part of W - int B over the ray is homeomorphic to a solid cylinder minus an open ball as is also shown in figure 4.

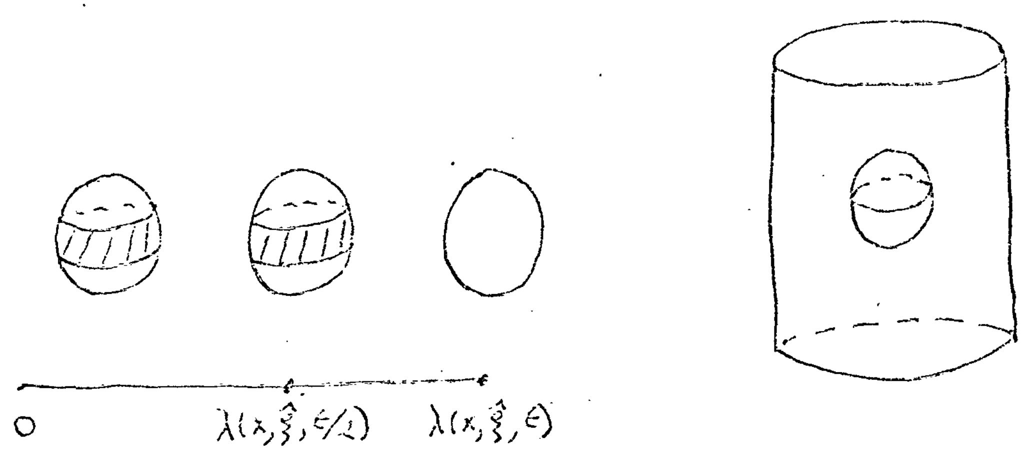

figure 4

Proof of proposition 5.8: Suppose $(\xi,x,\eta,y) \in W - \text{int } B$. If $0 < |\xi| \le \lambda(x,\hat{\xi},\epsilon/2)$ where $\hat{\xi} = \xi|\xi|^{-1}$, define $\sigma(\xi,x,\eta,y) = (|x|,\hat{\xi},\epsilon|\eta||y|^{-2}\eta,t)$ where

$$t = \begin{cases} 1+(1-|\xi|)[\lambda(x,\hat{\xi},\epsilon/2)]^{-1} & \text{if } \operatorname{sgn}(x\cdot y) = 1 \text{ and } |y|^2 \ge \epsilon|\eta|^2 \\ 1-(1-|\xi|)[\lambda(x,\hat{\xi},\epsilon/2)]^{-1} & \text{if } \operatorname{sgn}(x\cdot y) = -1 \text{ and } |y|^2 \ge \epsilon|\eta|^2 \\ \epsilon^{-1}|y|^2|\eta|^{-2}\operatorname{sgn}(x\cdot y) & \text{if } |\xi| = \lambda(x,\hat{\xi},\epsilon/2) \text{ and } |y|^2 \le \epsilon|\eta|^2 \end{cases}$$

We have seen previously that

$$Y = \{(\xi,x,\eta,y) \in W\text{-int } B\colon \lambda(x,\hat{\xi},\epsilon/2) \le |\xi| \le \lambda(x,\hat{\xi},\epsilon)\}$$

is homeomorphic to $R^1 \times S^1 \times S^2 \times [\lambda(x,\hat{\xi},\epsilon/2),\ \lambda(x,\hat{\xi},\epsilon)]$. Thus extend σ to map Y onto $R^1 \times S^1 \times X_1$. It can be shown that σ is the desired homeomorphism. Observe that if $(\xi,x,\eta,y) \in \partial B$ then $\sigma(\xi,x,\eta,y) = (|x|,\xi|\xi|^{-1},\eta|\eta|^{-1},t)$.

In order to construct the homeomorphism γ we need the following:

Lemma 5.10: Given $\delta > 0$, for $\epsilon > 0$ sufficiently small the map $\sigma\pi\sigma^{-1}$ is within δ of the map f. More precisely, if $(|x|,\theta,\varphi,t) \in \sigma(b^{+})$ where θ,φ are angle variables let

$$\sigma\pi\sigma^{-1}(|x|,\theta,\varphi,t) = (|x|_1,\theta_1,\varphi_1,t_1),$$

and let

$$f(|x|,\theta,\varphi,t) = (|x|_2,\theta_2,\varphi_2,t_2).$$

Then

$$||x_1|-|x_2|| < \delta|x|,\ |t_1-t_2| < \delta,\ |\theta_1-\theta_2| < \delta$$
$$\text{and } |\varphi_1-\varphi_2| < \delta.$$

Proof: For ϵ sufficiently small lemma 4.17 implies that

$$|x_1-x_2| < \delta_1|x|,\quad |y_1-y_2| < \delta_1|y|,$$
$$||\xi_1|-|\xi_2|| < \delta_1 \text{ and } ||\eta_1|-|\eta_2|| < \delta_1.$$

It follows that $|t_1-t_2| < \delta$ for small ϵ. (We use the fact that $\lambda(x,\hat{\xi},\epsilon)$ becomes less and less dependent on $\hat{\xi}$ as $\epsilon \to 0$.) Lemma 4.17 also implies that $|\theta_1-\theta_2| < \delta$ and $|\varphi_1-\varphi_2| < \delta$ for ϵ sufficiently small since σ "preserves" the angle variables θ,φ.

We now finish the proof of 5.8. It follows from 4.5 (a) and (b) and the definition of σ that $\sigma(\tau)$ is diffeomorphic and very close to $\mathscr{A}^+ \cap \mathscr{A}^-$. Thus we can choose a homeomorphism $\gamma_1: \mathscr{A}^+ \to \sigma(b^+)$ which is close to the identity. Define $\gamma_2: \sigma(b^-) \to \mathscr{A}^-$ by $\gamma_2 = \sigma\pi\sigma^{-1}\cdot\gamma_1\circ f^{-1}$. γ_2 is close to the identity by lemma 5.10. Then $\gamma_1 \cup \gamma_2$ is a homeomorphism of $R^1 \times S^1 \times S^1 \times R^1$ onto itself which is close to the identity. Furthermore the diagram

$$\begin{tikzcd} \mathscr{A}^+ \arrow[r, "f"] \arrow[d, "\gamma_1"'] & \mathscr{A}^- \arrow[d, "\gamma_2"] \\ \sigma(b^+) \arrow[r, "\sigma\pi\sigma^{-1}"] & \sigma(b^-) \end{tikzcd}$$

commutes. Since $\gamma_1 \cup \gamma_2$ is close to the identity, $\gamma_1 \cup \gamma_2$ extends to a homeomorphism γ of $R^1 \times S^1 \times X$ onto itself which is the identity outside a neighborhood of $R^1 \times S^1 \times S^1 \times R^1$. γ is the desired homeomorphism. This completes the proof.

6. The Isolating Block for Tripple Collision

The well known Lagrange-Jacobi identity can be used to show that there exists an isolating block such that any orbit which ends in a tripple collision must enter and remain in this block.

6.1: Define $I: R^8 - C \to R^1$ by

$$I(\xi,x,\eta,y) = \frac{2}{9}(3|\xi|^2 + 3|x|^2 - 4\xi \circ x).$$

In terms of the variables q_i and p_i, $I = |q_1|^2 + |q_2|^2 + |q_3|^2$. The Lagrange-Jacobi identity is

$$\ddot{I}(\xi,x,\eta,y) = U(\xi,x) + 2H(\xi,x,\eta,y) = U+2h.$$

Notice that $\gamma(t)$ is a solution of equations 3.7 which ends in tripple collision as $t \to t_0$ (i.e. $|\xi(t)| \to 0$, $|x(t)| \to 0$ as $t \to t_0$) if and only if $I(\gamma(t)) \to 0$ as $t \to t_0$.

6.2: Define $B_4[\epsilon] = \{(\xi,x,\eta,y): I(\xi,x,\eta,y) - \epsilon \leq 0\}$. Every solution which ends in tripple collision must eventually enter and remain in $B_4[\epsilon]$.

Proposition 6.3: For $\epsilon > 0$ sufficiently small $B_4[\epsilon]$ is an isolating block for the 3-body flow on $R^8 - C$.

Proof: Choose ϵ sufficiently small so that $I(\xi,x,\eta,y) \leq \epsilon$ implies that $U(\xi,x) + 2h > 0$. Then $\ddot{I} > 0$ on $B_4[\epsilon]$ and hence $B_4[\epsilon]$ is an isolating block.

Recall that tripple collision can only occur when the total angular momentum is zero. Hence consider the 3-body flow on $S[h,0]$. After regularization of binary collisions one obtains as before the regularized phase space $R[h,0]$ and an induced flow on $R[h,0]$. However this new flow still has a singularity due to tripple collisions. $B_4 = B_4[\epsilon] \cap S[h,0]$ can be considered as an isolating block for the flow in $R[h,0]$. C. Conley has shown that the flow map π_4: $b_4^+ - a_4^+ \to b_4^- - a_4^-$ does not admit a continuous extension as a map from b_4^+ onto b_4^-. Thus tripple collisions can not be geometrically regularized. This result compliments and in fact implies the classical result that solutions to equations 3.7 cannot in general be analytically continued beyond tripple collisions. Conley further shows that for each point (ξ,x) with $|\xi|^2 + |x|^2$ sufficiently small, there exists a choice of velocities (η_1,y_1) such that the orbit through the point (ξ,x,η_1,y_1) ends in tripple collision. These results depend on some facts about the topology of b_4^- and b_4^+. The following proposition topologically characterizes these sets.

Proposition 6.4:

(a) b_4 is homeomorphic to $(S^2\text{-}3D) \times S^2 \cup 3[R^1 \times P^3 - S^1 \times D^3]$ where $S^2\text{-}3D$ denotes the 2-sphere with three disks removed and where the boundary of $(S^2\text{-}3D) \times S^2$ is identified in the obvious way with the boundary of $3[R^1 \times P^3 - S^1 \times D^3]$.

(b) b_4^+ is homeomorphic to

$$(S^2-3D) \times D_- \cup 3[(-\infty,0] \times P^3 - S^1 \times D_-^3]$$

where D_- denotes the lower hemisphere of S^2, $[(-\infty,0] \times P^3 - S^1 \times D_-^3]$ denotes the space obtained from $S^1 \times X_-$ by identifying points via the map f as in 5. ($X_- = X \cap \{x \in R^3\colon x_3 \leq 0\}$). We identity $\{\partial(S^2-3D)\} \times D_-$ with $3[\partial(S^1 \times D_-^3)]$ where $\partial(S^1 \times D_-^3)$ denotes

$$S^1 \times \{x \in X\colon x_1^2+x_2^2 = \tfrac{1}{2} \text{ and } x_3 \leq 0\}.$$

<u>Sketch of the proof</u>: Using 5.8 choose $\gamma_i\colon (W_i - \text{int } B_i) \to R^1 \times [S^1 \times X]$ for $i = 1,2,3$ to be a homeomorphism such that the diagram

$$\begin{array}{ccc}
(W_i - \text{int } B_i) & \xrightarrow{\gamma_i} & R^1 \times [S^1 \times X] \\
\Big\downarrow i & & \Big\downarrow j \\
\hat{W}_i & \xrightarrow{\tilde{\gamma}_i} & R^1 \times [R^1 \times P^3 - S^1 \times D^3]
\end{array}$$

commutes where i and j are the natural projections and $\tilde{\gamma}_i$ is induced by γ_i. It can be shown that there exists a homeomorphism k of $R^1 \times (S^1 \times X)$ onto itself which is the identity on the factor $(S^1 \times X)$ and which takes $\gamma_i(b_4 \cap (W_i - \text{int } B_i))$ onto $(r) \times (S^1 \times X)$ for some $r \in R^1$. It follows that $j \cdot k \cdot \gamma_i\colon b_4 \cap (W_i - \text{int } B_i) \to (r) \times [R^1 \times P - S^1 \times D^3]$ is a homeomorphism. $(b_4 - \bigcup_{i=1}^{3} W_i)$ is easily seen to be homeomorphic to $(S^2-3D) \times S^2$. Assertion (a) follows since $b_4 = (b_4 - \bigcup_{i=1}^{3} W_i) \cup (b_4 \cap \bigcup_{i=1}^{3} W_i)$.

To prove assertion (b) one shows that $b_4^+ - \bigcup_{i=1}^{3} W_i$ is homeomorphic to $(S^2-3D) \times D_-$. The homeomorphism $j \cdot k \cdot \gamma_i$: $b_4^+ \cap (W_i - \operatorname{int} B_i) \to R^1 \times [R^1 \times P^3 - S^1 \times D^3]$ takes $b_4^+ \cap (W_i - \operatorname{int} B_i)$ into $(r) \times [(-\infty,0] \times P^3 - S^1 \times D_-^3]$. Assertion (b) follows since

$$b_4^+ = (b_4^+ - \bigcup_{i=1}^{3} W_i) \cup (b_4^+ \cap \bigcup_{i=1}^{3} W_i).$$

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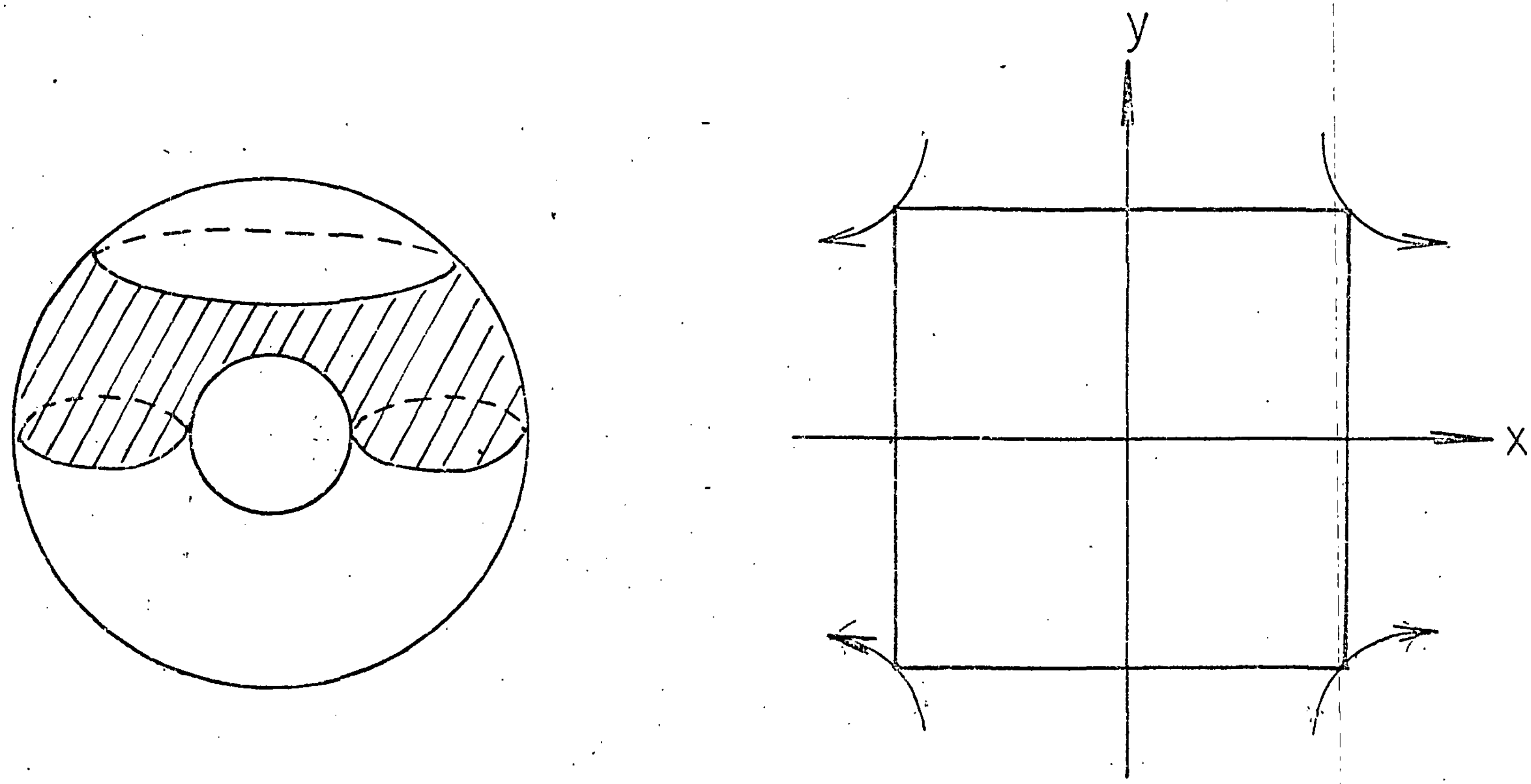

FIG. I

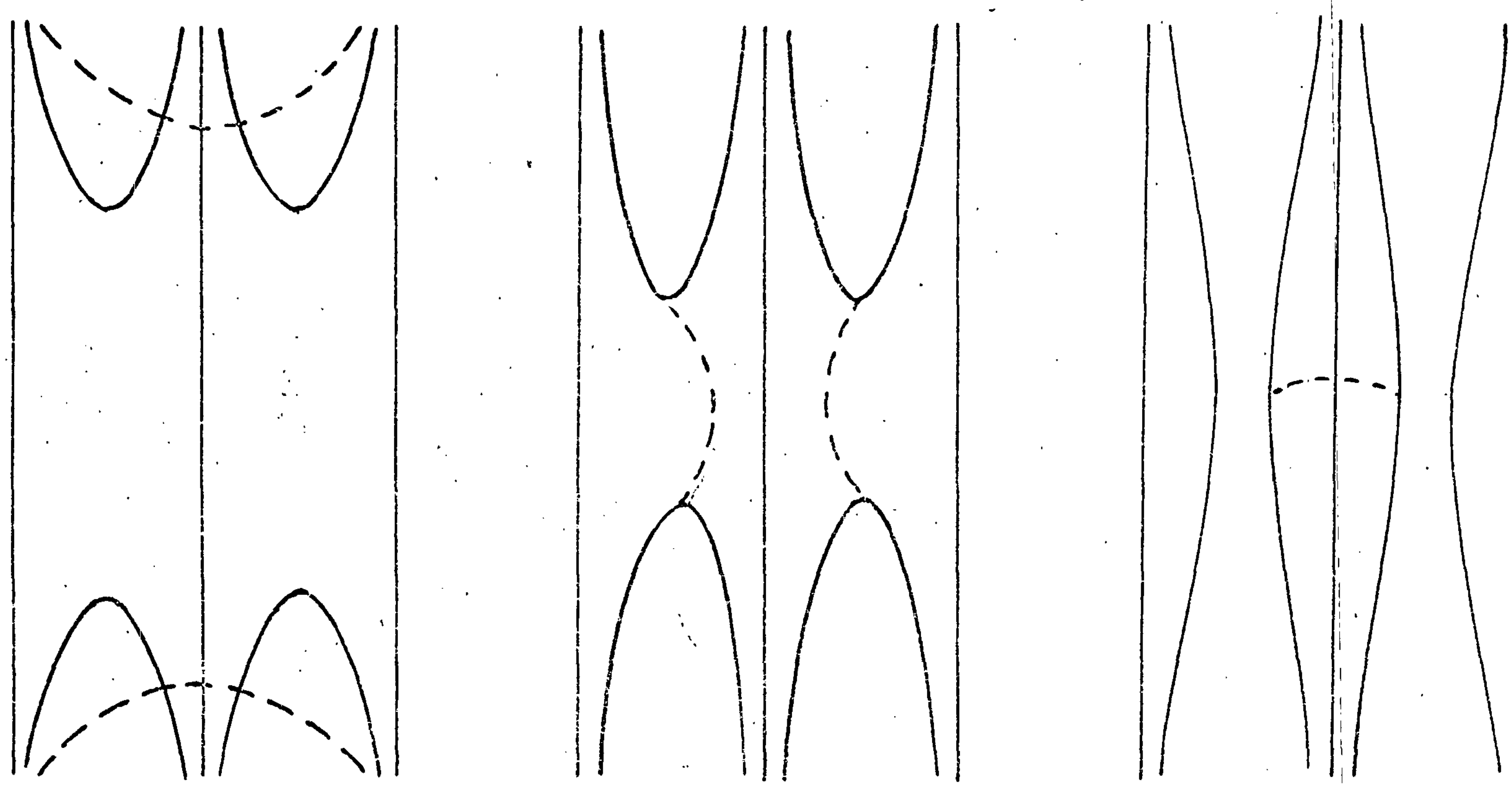

FIG 2

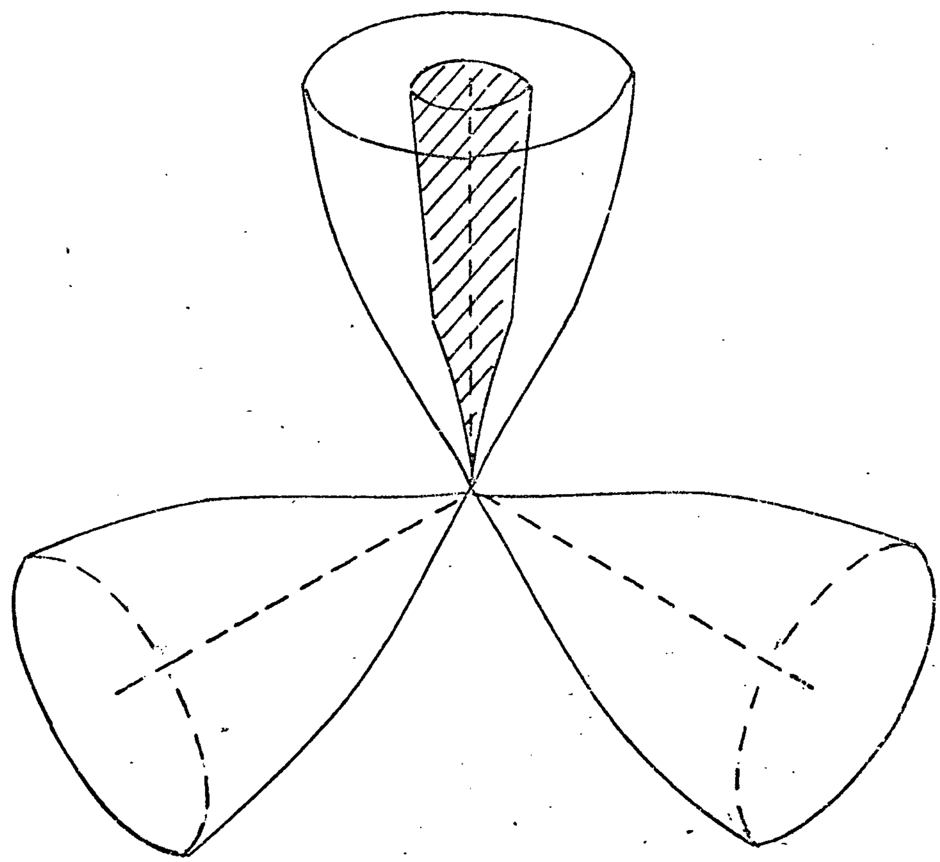

FIG. 3

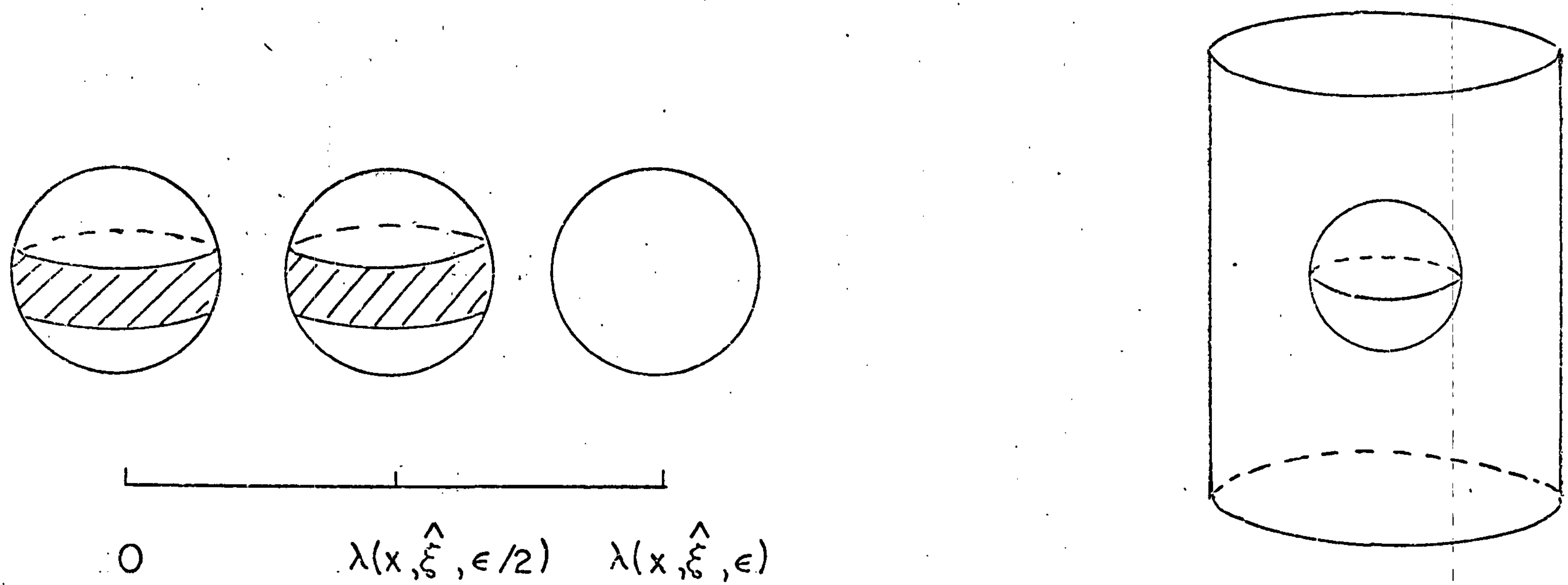

FIG. 4